# Classification of transversal Lagrangian stars

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**Abstract**

A Lagrangian star is a system of three Lagrangian submanifolds of the symplectic space intersecting at a common point. In this work we classify transversal Lagrangian stars in the symplectic space in the analytic category under the action of symplectomorphisms by using the method of algebraic restrictions. We present a list of all transversal Lagrangian star.

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## 1. Introduction

The problem of classification of germs of $s$ Lagrangian submanifolds $L_1, \cdots, L_s$ intersecting at a common point $p$ (defined in [J] as $s$-Lagrangian star at $p$) under the action of symplectomorphisms was introduced by Janeczko in [J]. In the case of three Lagrangian subspaces in a symplectic vector space $(M, \omega)$ under the action of symplectic transformations, the natural invariant is the Maslov index ([LV]), that is, the signature of the Kashiwara quadratic form $Q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) +$

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$\omega(x_3, x_1)$ defined on the direct sum of the Lagrangian subspaces. Janeczko generalizes the Maslov index to the nonlinear case.

The aim of this paper is to obtain the symplectic classification of 3-Lagrangian stars two by two transversal in a symplectic space. For this purpose we use the method of algebraic restrictions introduced in [DJZ2]. We obtain a list of all transversal Lagrangian star.

A generalization of the Darboux-Givental Theorem ([AG]) to germs of quasi-homogeneous subsets of the symplectic space was obtained in [DJZ2] and reduces the problem of symplectic classification of germs of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these subsets. By this method, complete symplectic classifications of the $A-D-E$ singularities of planar curves and the $S_5$ singularity were obtained in [DJZ2].

The method of algebraic restrictions was used to study the local symplectic algebra of 1-dimensional singular analytic varieties. It is proved in [D1] that the vector space of algebraic restrictions of closed 2-forms to a germ of 1-dimensional singular analytic variety is a finite-dimensional vector space.

The method of algebraic restrictions was also applied to the zero-dimensional symplectic isolated complete intersection singularities (see [D2]) and to other 1-dimensional isolated complete intersection singularities: the $S_\mu$ symplectic singularities for $\mu > 5$ in [DT1], the $T_7 - T_8$ symplectic singularities in [DT2], the $W_8 - W_9$ symplectic singularities in [T1] and the $U_7, U_8$ and $U_9$ symplectic singularities in [T2]. In [DJZ3] the method is used to construct a complete system of invariants in the problem of classifying singularities of immersed $k$-dimensional submanifolds of a symplectic $2n$-manifold at a generic double point. In [ADW], the authors studied the local symplectic algebra of curves with semigroup $(4, 5, 6, 7)$ by this method.

This paper is organized as follows. Section 2 contains basic definitions about Lagrangian stars and the formulation of the main result. We also explain why we use the method of algebraic restrictions for this problem. We recall the method of algebraic restrictions in Section 3. In Section 4 we reduce the problem of classification of algebraic restrictions of symplectic forms to the linear case. Finally in Section 5 we obtain the symplectic classification of 3-Lagrangian stars two by two transversal.

## 2. Lagrangian stars

Consider $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$ the $2n$-dimensional symplectic space with coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Let $\{L_1, ..., L_s\}$ be a system of Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega)$ intersecting at the origin.

**Definition 2.1** ([J])**.** *The germ of Lagrangian submanifolds* $(\{L_1,\dots,L_s\},0)$ *is called* $s$***-Lagrangian star****. If* $s=2$ *and* $L_1$ *is transversal to* $L_2$ *then the 2-Lagrangian star* $(\{L_1,L_2\},0)$ *is called the basic Lagrangian star. The 3-Lagrangian star is simply called a Lagrangian star. We denote* $L=L_1\cup\cdots\cup L_s$*.*

**Definition 2.2.** *The germ of a subset* $N\subset(\mathbb{R}^m,0)$ *is called quasi-homogeneous if there exist a local coordinate system* $x_1,\dots,x_m$ *of* $(\mathbb{R}^m,0)$ *and positive integers* $\lambda_1,\dots,\lambda_m$ *with the following property: if* $(a_1,\dots,a_m)\in N$ *then* $(t^{\lambda_1}a_1,\dots,t^{\lambda_m}a_m)\in N$*, for all* $t\in[0,1]$*. The integers* $\lambda_1,\dots,\lambda_m$ *are called weights of the variables* $x_1,\dots,x_m$*, respectively.*

Let $E=(\{L_1,\dots,L_s\},0)$ be an $s$-Lagrangian star. We call $E$ a *quasi-homogeneous* $s$-Lagrangian star if $L=L_1\cup\cdots\cup L_s$ is a germ of a quasi-homogeneous subset. Moreover, $E$ is called transversal if $L_1,\dots,L_s$ are two by two transversal intersecting only at the origin.

Given $E=(\{L_1,\dots,L_s\},0)$ and $E'=(\{L'_1,\dots,L'_s\},0)$ two $s$-Lagrangian stars we say that they are diffeomorphic if there exists a germ of diffeomorphism $\Phi:(\mathbb{R}^{2n},0)\to(\mathbb{R}^{2n},0)$ such that $\Phi(L_i)=L^{'}_{j_i}$ for some permutation $j_i$ of $\{1,\dots,s\}$. When $\Phi$ is a germ of a symplectomorphism of $((\mathbb{R}^{2n},\omega),0)$ we say that $E$ and $E'$ are *symplectically equivalent* (or equivalent).

The germ of a Langrangian submanifold of $(\mathbb{R}^{2n},\sum_{i=1}^n dx_i\wedge dx_i)$ is symplectically equivalent to $L_1=\{(x,y)\in\mathbb{R}^{2n}|x_1=\cdots=x_n=0\}$. The germ $L_2$ at 0 of a Langrangian submanifold of $(\mathbb{R}^{2n},\sum_{i=1}^n dx_i\wedge dx_i)$ which is transversal to $L_1$ at 0 can be desribed in the following way

$$y_i=\frac{\partial S}{\partial x_i}(x_1,\cdots,x_n)\ \text{ for } r=i,\cdots,n,$$

where $S$ is a smooth function-germ on $\mathbb{R}^n$. Thus the transversal Lagrangian 2-star is symplectically equivalent to the basic Lagrangian star $(\{L_1,L_2\},0)$ defined by $L_1=\{(x,y)\in\mathbb{R}^{2n}|x_1=\cdots=x_n=0\}$ and $L_2=\{(x,y)\in\mathbb{R}^{2n}|y_1=\cdots=y_n=0\}$, by a symplectomorphism of the following form

$$\Phi:\mathbb{R}^{2n}\ni(x,y)\mapsto\left(x_1,\cdots,x_n,y_1-\frac{\partial S}{\partial x_1}(x_1,\cdots,x_n),\cdots,y_n-\frac{\partial S}{\partial x_n}(x_1,\cdots,x_n)\right).$$

It implies that a trasversal Lagrangian 3-star is symplectically equivalent to a Lagrangian 3-star $(\{L_1,L_2,L_3\},0)$, where $L_1=\{(x,y)\in\mathbb{R}^{2n}|x_1=\cdots=x_n=0\}$, $L_2=\{(x,y)\in\mathbb{R}^{2n}|y_1=\cdots=y_n=0\}$ and $L_3$ can be desribed in the following way

$$y_i=\frac{\partial S}{\partial x_i}(x_1,\cdots,x_n)\ \text{ for } r=i,\cdots,n,$$

where $S$ is a smooth function-germ on $\mathbb{R}^n$. Using the classical method for the classification of transversal Lagrangian 3-star $(\{L_1, L_2, L_3\}, 0)$ we should apply the symplectomorphisms which preserve the set $L_1 \cup L_2$ to obtain the normal form of $L_3$. It is easy to see that such symplectomorphisms have following forms $\Phi(x,y) = (\Phi_1(x,y), \Phi_2(x,y))$ or $\Psi(x,y) = (\Psi_1(x,y), \Psi_2(x,y))$, where $\Phi_i, \Psi_i : \mathbb{R}^{2n} \to \mathbb{R}^n$ for $i = 1, 2$ such that $\Phi_1(0,y) = \Phi_2(x,0) = \Psi_1(x,0) = \Psi_2(0,y) = 0$. A Hamiltonian vector field $X_H = \sum_{i=1}^n \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i}$ is tangent to $L_1 \cup L_2$ if the Hamiltonian function-germ $H$ satisfies the following system of equations $y_j \frac{\partial H}{\partial y_i} - x_i \frac{\partial H}{\partial x_j} = \sum_{k=1}^n \sum_{l=1}^n x_k y_l g_{i,j,k,l}(x,y)$ for $i, j = 1, \cdots, n.$, where $g_{i,j,k,l}$ are function-germs on $\mathbb{R}^{2n}$. Hamiltonian function-germs of the form $H(x,y) = \sum_{i=1}^n \sum_{j=1}^n x_i y_j f_{i,j}(x,y)$, where $f_{i,j}$ are function-germs on $\mathbb{R}^{2n}$, satisfy the above system of equations. So the classical method is complicated for trasversal Lagrangian 3-stars. Therefore we apply the method of algebraic restriction to obtain the following classification theorem, which is the main result of this paper.

**Theorem 2.3.** *A transversal Lagrangian* 3*-star in* $(\mathbb{R}^{2n}, \sum_{i=1}^n dx_i \wedge dy_i)$ *is symplectically equivalent to one and only one of* $E^s = (\{L_1, L_2, L_3^s\}, 0)$, *where*

$$L_1 = \{x_1 = \cdots = x_n = 0\}, \quad L_2 = \{y_1 = \cdots = y_n = 0\},$$

$$L_3^s = \{x_1 - y_1 = \cdots = x_s - y_s = x_{s+1} + y_{s+1} = \cdots = x_n + y_n = 0\},$$

*and* $s$ *is a non-negative integer such that* $s \le \frac{n}{2}$.

**Notations:** Let $\theta$ be a $k$-form on $((\mathbb{R}^{2n}, \omega), 0)$ and let $E = (\{L_1, \ldots, L_s\}, 0)$ be a $s$-Lagrangian star.

1. The set of smooth points of $L$ is denoted by $L_{\mathrm{reg}}$.
2. The restriction of $\theta$ to the set $\{(p, v_1, \ldots, v_k) | p \in L_j \text{ and } v_1, \ldots, v_k \in T_p L_j\}$ is denoted by $\theta|_{TL_j}$, $j = 1, \ldots, s$.
3. Suppose $\theta(p)(u_1, \ldots, u_k) = 0$ for every $p \in L_{\mathrm{reg}}$ and $u_1, \ldots, u_k \in T_p L_{\mathrm{reg}}$. In this case, we say that $\theta$ vanish on $TL_{\mathrm{reg}}$.

All objects in this paper (functions, vector fields, $k$-forms, maps) are $\mathbb{R}$-analytic.

## 3. Method of algebraic restrictions

In this section we present the method of algebraic restrictions. More details can be found in [DJZ2].

Let $M$ be a germ of smooth manifold. We denote by $\Lambda^k(M)$ the space of all germs at 0 of differential $k$-forms on $M$. Given a subset $N \subset M$ one introduces the following subspaces of $\Lambda^k(M)$:

$$\Lambda^k_N(M) = \{\omega \in \Lambda^k(M) : \omega(x) = 0, \text{ for all } x \in N\},$$
$$\mathscr{A}^k_0(N, M) = \{\alpha + d\beta : \alpha \in \Lambda^k_N(M), \beta \in \Lambda^{k-1}_N(M)\}.$$

The notation $\omega(x) = 0$ means that the $k$-linear form $\omega(x)$ vanishes for all $k$-tuple of vectors in $T_xM$, i. e. all coefficients of $\omega$ in some (and then any) local coordinate system vanish at the point $x$.

**Definition 3.1** ([DJZ2])**.** *Let $N$ be a subset of $M$ and let $\theta \in \Lambda^k(M)$. The **algebraic restriction** of $\theta$ to $N$ is the equivalence class of $\theta$ in $\Lambda^k(M)$, where the equivalence is as follows: $\theta$ is equivalent to $\tilde{\theta}$ if $\theta - \tilde{\theta} \in \mathscr{A}^k_0(N, M)$. The algebraic restriction of $\theta$ to $N$ is denoted by $[\theta]_N$.*

**Notation:** Let $\theta$ be a $k$-form on $M$. Writing $[\theta]_N = 0$ (or saying that $\theta$ has zero algebraic restriction to $N$) we mean that $[\theta]_N = [0]_N$, i.e. $\theta \in \mathscr{A}^k_0(N, M)$.

**Remark 3.2.** *It is clear that if $\theta \in \mathscr{A}^k_0(N, M)$ then $d\theta \in \mathscr{A}^{k+1}_0(N, M)$. Moreover, if $\theta_1$ is a $k$-form such that $[\theta_1]_N = 0$ then $[\theta_1 \wedge \theta_2]_N = 0$ for every $q$-form $\theta_2$. Then if $\theta_1$ is a $k$-form and if $\theta_2$ is a $q$-form the algebraic restrictions $d[\theta_1]_N := [d\theta_1]_N$ and $[\theta_1]_N \wedge [\theta_2]_N := [\theta_1 \wedge \theta_2]_N$ are well defined.*

Let $M$ and $\tilde{M}$ be manifolds and $\Phi : \tilde{M} \to M$ a local diffeomorphism. Let $N$ be a subset of $M$. It is clear that $\Phi^*\mathscr{A}^k_0(N, M) = \mathscr{A}^k_0(\Phi^{-1}(N), \tilde{M})$. Therefore the action of the group of diffeomorphisms can be defined as follows: $\Phi^*([\theta]_N) := [\Phi^*\theta]_{\Phi^{-1}(N)}$, where $\theta$ is an arbitrary $k$-form on $M$. Let $\tilde{N} \subset \tilde{M}$. Two algebraic restrictions $[\theta]_N$ and $[\tilde{\theta}]_{\tilde{N}}$ are called **diffeomorphic** if there exists a local diffeomorphism form $\tilde{M}$ to $M$ sending one algebraic restriction to another. This of course requires that the diffeomorphism sends $\tilde{N}$ to $N$. If $M = \tilde{M}$ and $N = \tilde{N}$, $\Phi$ is called a local symmetry of $N$.

The method of algebraic restrictions is based on the following result:

**Theorem 3.3.** (i) (Theorem A in [DJZ2]) *Let $N$ be a quasi-homogeneous subset of $\mathbb{R}^{2n}$. Let $\omega_0, \omega_1$ be symplectic forms on $\mathbb{R}^{2n}$ with the same algebraic restriction to $N$. There exists a local diffeomorphism $\Phi$ such that $\Phi(x) = x$ for any $x \in N$ and $\Phi^*\omega_1 = \omega_0$.*

(ii) (Corollary of (i)) *Let $\tilde{E} = (\{\tilde{L_1}, \ldots, \tilde{L_s}\}, 0)$ and $\hat{E} = (\{\hat{L_1}, \ldots, \hat{L_s}\}, 0)$ be $s$-Lagrangian stars diffeomorphic to a quasi-homogeneous $s$-Lagrangian star $E = (\{L_1, \ldots, L_s\}, 0)$. Then $\tilde{E}$ and $\hat{E}$ are equivalents if and only if $[\omega]_{\hat{L}}$ and $[\omega]_{\tilde{L}}$ are diffeomorphic.*

**Remark 3.4.** (i) *Let $E = (\{L_1, L_2, L_3\}, 0)$ be a transversal quasi-homogeneous Lagrangian star. Due to Theorem 3.3, the symplectic classification of transversal Lagrangian stars diffeomorphic to $E$ reduces to the classification of algebraic restrictions of symplectic forms to $L$ vanishing on $TL_{\mathrm{reg}}$.*

(ii) *Let $\tilde{E} = (\{\tilde{L}_1, \tilde{L}_2, \tilde{L}_3\}, 0)$ be a transversal Lagrangian star in $((\mathbb{R}^{2n}, \omega), 0)$. It is not difficult to prove that there exists a smooth coordinate change in $(\mathbb{R}^{2n}, 0)$ such that, for all $i$, $\tilde{L}_i = L_i$, where $L_1 = \{y_1 = \cdots = y_n = 0\}$, $L_2 = \{x_1 = \cdots = x_n = 0\}$ and $L_3 = \{x_1 - y_1 = \cdots = x_n - y_n = 0\}$.*

**Definition 3.5.** *The germ of a function, of a differential $k$-form, or of a vector field $\alpha$ on $(\mathbb{R}^m, 0)$ is **quasi-homogeneous** in a coordinate system $(x_1, \ldots, x_m)$ on $(\mathbb{R}^m, 0)$ with positive weights $(\lambda_1, \ldots, \lambda_m)$ if $\mathcal{L}_E \alpha = \delta \alpha$, where $E = \sum_{i=1}^m \lambda_i x_i \partial/\partial x_i$ is the germ of the Euler vector field on $(\mathbb{R}^m, 0)$ and $\delta$ is a real number called the **quasi-degree**.*

It is easy to show that $\alpha$ is quasi-homogeneous in a coordinate system $(x_1, \ldots, x_m)$ with weights $(\lambda_1, \ldots, \lambda_m)$ if and only if $F_t^* \alpha = t^\delta \alpha$, where $F_t(x_1, \ldots, x_m) = (t^{\lambda_1} x_1, \ldots, t^{\lambda_m} x_m)$. Thus germs of quasi-homogeneous functions of quasi-degree $\delta$ are germs of weighted homogeneous polynomials of degree $\delta$. The coefficient $f_{i_1, \ldots, i_k}$ of the quasi-homogeneous differential $k$-form $\sum f_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ of quasi-degree $\delta$ is a weighted homogeneous polynomial of degree $\delta - \sum_{j=1}^k \lambda_{i_j}$. The coefficient $f_i$ of the quasi-homogeneous vector field $\sum_{i=1}^m f_i \partial/\partial x_i$ of quasi-degree $\delta$ is a weighted homogeneous polynomial of degree $\delta + \lambda_i$.

Let $\theta$ be the germ of a $k$-form on $(\mathbb{R}^m, 0)$. We denote by $\theta^{(r)}$ the quasi-homogeneous part of quasi-degree $r$ in the Taylor series of $\theta$. It is clear that if a smooth function $h$ vanishes on a quasi-homogeneous set $N$ then $h^{(r)}$ also vanishes on $N$, for every non-negative $r$. This simple observation implies the following result:

**Proposition 3.6.** *If $\theta$ is a $k$-form on $(\mathbb{R}^m, 0)$ with $[\theta]_N = 0$ then $[\theta^{(r)}]_N = 0$, for any $r$.*

Proposition 3.6 allows us to define the quasi-homogeneous part of an algebraic restriction.

**Definition 3.7.** *Let $a = [\theta]_N$ be an algebraic restriction of a $k$-form $\theta$ to a germ of quasi-homogeneous subset $N \subset (\mathbb{R}^m, 0)$. We define the quasi-homogeneous part of quasi-degree $r$ of $a$ by $a^{(r)} := [\theta^{(r)}]_N$. When $a = a^{(r)}$ we say that $a$ is quasi-homogeneous of quasi-degree $r$.*

**Proposition 3.8** ([D1])**.** *If $X$ is the germ of a quasi-homogeneous vector field of quasi-degree $i$ and $\omega$ is the germ of a quasi-homogeneous differential form of quasi-degree $j$ then $\mathcal{L}_X \omega$ is the germ of a quasi-homogeneous differential form of quasi-degree $i + j$.*

Throughout this paper, $E = (\{L_1, L_2, L_3\}, 0)$ is the Lagrangian star in $((\mathbb{R}^{2n}, \omega), 0)$ where $L_1 = \{y_1 = \cdots = y_n = 0\}$, $L_2 = \{x_1 = \cdots = x_n = 0\}$ and $L_3 = \{x_1 - y_1 = \cdots = x_n - y_n = 0\}$.

Let $k$ be a non-negative integer. Let us fix some notations:

- $\Lambda^k_{\mathrm{reg}}(\mathbb{R}^{2n})$: the vector space of the $k$-forms vanishing on $TL_{\mathrm{reg}}$.
- $[\Lambda^k_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$: the vector space of algebraic restrictions to $L$ of elements of $\Lambda^k_{\mathrm{reg}}(\mathbb{R}^{2n})$.
- $\Lambda^{k,cl}_{\mathrm{reg}}(\mathbb{R}^{2n})$: the subspace of $\Lambda^k_{\mathrm{reg}}(\mathbb{R}^{2n})$ consisting of closed $k$-forms vanishing on $TL_{\mathrm{reg}}$.
- $[\Lambda^{k,cl}_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$: the subspace of $[\Lambda^k_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$ consisting of algebraic restrictions to $L$ of elements of $\Lambda^{k,\mathrm{cl}}_{\mathrm{reg}}(\mathbb{R}^{2n})$.

## 4. Reduction to the linear case

In this section we reduce the classification of the algebraic restrictions to $L$ of symplectic forms vanishing on $TL_{\mathrm{reg}}$ under the action of local symmetries of $L$ to the classification of the algebraic restrictions to $L$ of homogeneous symplectic forms of degree 2 under the action of linear local symmetries of $L$.

The first step is to find a finite set of generators of $[\Lambda^{2,\mathrm{cl}}_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$. For this we need some results.

**Proposition 4.1.** *Let $\theta$ be a $k$-form in $\Lambda^k_{\mathrm{reg}}(\mathbb{R}^{2n})$. Then $\theta^{(r)} \in \Lambda^k_{\mathrm{reg}}(\mathbb{R}^{2n})$ , for all non-negative integer $r \geq k$.*

*Proof.* Since $\theta$ is a $k$-form then $\theta^{(r)} = 0$ for all $r = 0, \ldots, k-1$. Let $r \geq k$. Writing $\theta$ in its Taylor series we have

$$\theta = \theta^{(k)} + \cdots + \theta^{(r)} + T,$$

where $\theta^{(s)}$ is a homogeneous $k$-form of degree $s$, $s = k, \ldots, r$, and $T$ is a $k$-form with $T^{(i)} = 0$, $i = 0, \ldots, r$.

Let $p \in L_j$ and $u_1, \ldots, u_k \in T_pL_j = L_j$, for some $j \in \{1, 2, 3\}$. Let $u = (u_1, \ldots, u_k)$ then for $t \neq 0$ small enough we have

$$0 = \theta(tp)u = \theta^{(k)}(p)u + \cdots + t^{r-k}\theta^{(r)}(p)u + T(tp)u.$$

Since $\lim_{t\to 0} \dfrac{T(tp)}{t^{r-k}} = 0$ we conclude that $\theta^{(r)}$ vanishes on $TL_j$, for all $j \in \{1, 2, 3\}$. Therefore $\theta^{(r)}$ vanishes on $TL_{\mathrm{reg}}$. $\square$

Next we show that the generators of $[\Lambda^{2,\mathrm{cl}}_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$ are obtained as derivatives of the generators of $[\Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$. In this case we reduce our problem to 1-forms.

**Proposition 4.2.** *Let $\sigma$ be a 2-form in $\Lambda^{2,cl}_{\mathrm{reg}}(\mathbb{R}^{2n})$. Then there exists a 1-form $\gamma$ in $\Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})$ such that $\sigma = d\gamma$.*

*Proof.* We use the method described in [DJZ1]. Define $F : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by $F(t,x,y) = F_t(x,y) = (tx,ty)$. Let $X_t$ be the vector field associated with the equation $\frac{dF_t}{dt} = X_t \circ F_t$, for $0 < t_0 \leq t \leq 1$. We have

$$\sigma - {F_{t_0}}^*\sigma = \int_{t_0}^1 \frac{d}{dt} F_t^*\sigma dt = \int_{t_0}^1 F_t^*(\mathcal{L}_{X_t}\sigma)dt = \int_{t_0}^1 F_t^*(d(i_{X_t}\sigma))dt = d\int_{t_0}^1 F_t^*(i_{X_t}\sigma)dt.$$

Letting $t_0 \to 0$ we get $\sigma = d\beta$ where $\beta = \int_0^1 F_t^*(i_{X_t}\sigma)dt$. For every $p \in L_{\mathrm{reg}}$ and $v \in T_pL_{\mathrm{reg}}$ we have

$$F_t^*(i_{X_t}\sigma)(p)\cdot v = \sigma(tp)(X_t \circ F_t(p), dF_t(p)\cdot v) = 0,$$

for all $t \in (0,1]$. Then $\beta(p)\cdot v = 0$. □

**Proposition 4.3.** *If $\gamma \in \Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})$ then $d\gamma \in \Lambda^{2,cl}_{\mathrm{reg}}(\mathbb{R}^{2n})$.*

*Proof.* We can write $\gamma = \sum_{j=1}^n (f_j dx_j + g_j dy_j)$, where $f_j$ and $g_j$ are germs of functions on $(\mathbb{R}^{2n},0)$, $j = 1,\ldots,n$. We have

$$d\gamma = \sum_{i,j=1}^n \left( \left( \frac{\partial f_j}{\partial x_i} dx_i + \frac{\partial f_j}{\partial y_i} dy_i \right) \wedge dx_j + \left( \frac{\partial g_j}{\partial x_i} dx_i + \frac{\partial g_j}{\partial y_i} dy_i \right) \wedge dy_j \right).$$

As $\gamma|_{TL_{\mathrm{reg}}} = 0$ then $f_j(x,0) = 0$, $g_j(0,y) = 0$ and $(f_j + g_j)(z,z) = 0$. Thus $d\gamma|_{TL_1} = d\gamma|_{TL_2} = 0$ and

$$d\gamma|_{TL_3} = \sum_{i,j=1}^n \left( \left( \frac{\partial f_j}{\partial z_i}(z,z) + \frac{\partial g_j}{\partial z_i}(z,z) \right) dz_i \wedge dz_j \right) = 0.$$

□

Due to Proposition 4.1, $\Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})$ is generated by homogeneous 1-forms. Next we find generators of $\Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})$ homogeneous of degree $\leq 4$. In Lemma 4.4, we prove that the elements in $\Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})$ homogeneous of degree $\geq 5$ have zero algebraic restriction to $L$. We conclude that $[\Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$ is a finite dimensional vector space generated by algebraic restrictions to $L$ of homogeneous 1-forms vanishing on $TL_{\text{reg}}$ of degree $\leq 4$.

Clearly there is no homogeneous 1-forms of degree 1 in $\Lambda^1_{\text{reg}}(\mathbb{R}^{2n})$.

**Generators of degree 2:**

Let $\gamma = \sum_{i,j=1}^{n}(a_{ij}x_i dx_j + b_{ij}x_i dy_j + c_{ij}y_i dx_j + e_{ij}y_i dy_j)$ be a 1-form in $\Lambda^1_{\text{reg}}(\mathbb{R}^{2n})$. It is easy to see that $\sum_{i,j=1}^{n} a_{ij}x_i dx_j = \sum_{i,j=1}^{n} e_{ij}y_i dy_j = 0$ and $b_{ij} = -c_{ji}$, for all $i,j \in \{1,\dots,n\}$. Thus the homogeneous 1-forms of degree 2 in $\Lambda^1_{\text{reg}}(\mathbb{R}^{2n})$ are linear combination of the 1-forms:

- $x_i dy_j - y_i dx_j, \;\; i,j \in \{1,\dots,n\}$.

Analogously we find the generators of degree 3 and 4.

**Generators of degree 3:**

- $x_i x_j dy_k - x_i y_j dy_k$
- $x_i x_j dy_k - y_i x_j dy_k$
- $x_i x_j dy_k - x_i y_j dx_k$
- $x_i x_j dy_k - y_i x_j dx_k$
- $x_i x_j dy_k - y_i y_j dx_k$

where $i,j,k \in \{1,\dots,n\}$.

**Generators of degree 4:**

- $x_i x_j x_k dy_l - x_i x_j y_k dy_l$
- $x_i x_j x_k dy_l - x_i y_j x_k dy_l$
- $x_i x_j x_k dy_l - y_i x_j x_k dy_l$
- $x_i x_j x_k dy_l - x_i y_j y_k dy_l$
- $x_i x_j x_k dy_l - y_i x_j y_k dy_l$
- $x_i x_j x_k dy_l - y_i y_j x_k dy_l$
- $x_i x_j x_k dy_l - x_i x_j y_k dx_l$
- $x_i x_j x_k dy_l - x_i y_j x_k dx_l$
- $x_i x_j x_k dy_l - y_i x_j x_k dx_l$
- $x_i x_j x_k dy_l - x_i y_j y_k dx_l$
- $x_i x_j x_k dy_l - y_i x_j y_k dx_l$
- $x_i x_j x_k dy_l - y_i y_j x_k dx_l$
- $x_i x_j x_k dy_l - y_i y_j y_k dx_l$

where $i,j,k,l \in \{1,\dots,n\}$.

**Lemma 4.4.** *The 1-forms homogeneous of degree greater than or equal to 5 in $\Lambda^1_{\text{reg}}(\mathbb{R}^{2n})$ have zero algebraic restriction to $L$.*

*Proof.* Let $\tilde{\gamma} = \sum_{j=1}^{n}(f_j(x,y)dx_j + g_j(x,y)dy_j)$ in $\Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})$, where $f_j, g_j$ are germs of functions on $(\mathbb{R}^{2n},0)$, $i=1,\ldots,n$. As $\tilde{\gamma}$ vanishes on $TL_1$ and $TL_2$ then $\tilde{\gamma} = \sum_{i,j=1}^{n}(y_i f_{ij}(x,y)dx_j + x_i g_{ij}(x,y)dy_j)$, where $f_{ij}, g_{ij}$ are germs of functions on $(\mathbb{R}^{2n},0)$, $i,j=1,\ldots,n$.

Let $\gamma$ be a homogeneous 1-form of degree $l+1 \geq 5$ in $\Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})$, then $\gamma$ is of the form:

$$\begin{aligned}\gamma = \quad & \sum a_{1,i_1\cdots i_l k} y_{i_1} x_{i_2}\cdots x_{i_l} dx_k + \cdots + \sum a_{l,i_1\cdots i_l k} y_{i_1}\cdots y_{i_l} dx_k + \\ & \sum b_{1,i_1\cdots i_l k} x_{i_1} y_{i_2}\cdots y_{i_l} dy_k + \cdots + \sum b_{l,i_1\cdots i_l k} x_{i_1}\cdots x_{i_l} dy_k,\end{aligned}$$

where $i_1,\ldots,i_l,k \in \{1,\ldots,n\}$. As $\gamma|_{TL_3} = 0$, for $i_1,\ldots,i_l,k \in \{1,\ldots,n\}$ one has

$$\sum_{\sigma\in S_l}(a_{1,\sigma(i_1)\cdots\sigma(i_l)k} + \cdots + a_{l,\sigma(i_1)\cdots\sigma(i_l)k} + b_{1,\sigma(i_1)\cdots\sigma(i_l)k} + \cdots + b_{l,\sigma(i_1)\cdots\sigma(i_l)k}) = 0,$$

where $S_l$ is the group of permutation of $\{i_1,\ldots,i_l\}$. Thus the 1-forms homogeneous of degree $l+1$ in $\Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})$ are generated by 1-forms of the type

$$\rho_{t\sigma} = y_{i_1}y_{i_2}x_{i_3}\cdots x_{i_l}dx_k - x_{\sigma(i_1)}\cdots x_{\sigma(i_t)}y_{\sigma(i_{t+1})}\cdots y_{\sigma(i_l)}dx_k,$$

$$\xi_{t\sigma} = y_{i_1}y_{i_2}x_{i_3}\cdots x_{i_l}dx_k - y_{\sigma(i_1)}\cdots y_{\sigma(i_t)}x_{\sigma(i_{t+1})}\cdots x_{\sigma(i_l)}dy_k,$$

where $\sigma \in S_l$ and $0 \leq t \leq l-1$.

Let $t \in \{1,\ldots,l-1\}$. Observe that the polynomial $y_{i_1}y_{i_2}x_{i_3}\cdots x_{i_l}x_k - x_{\sigma(i_1)}\cdots x_{\sigma(i_t)}y_{\sigma(i_{t+1})}\cdots y_{\sigma(i_l)}x_k$ vanishes on $L$. Then the 1-forms of the type $\rho_{t\sigma}$ and $\xi_{t\sigma}$ are generated by

$$\rho = y_{i_1}y_{i_2}x_{i_3}\cdots x_{i_l}dx_k - y_{i_1}\cdots y_{i_l}dx_k$$

$$\xi_1 = y_{i_1}y_{i_2}x_{i_3}\cdots x_{i_l}dx_k - x_{i_1}\cdots x_{i_l}dy_k$$

$$\xi_2 = y_{i_1}y_{i_2}x_{i_3}\cdots x_{i_l}(dx_k - dy_k).$$

Note that the polynomial $h(x,y) = y_{i_1}y_{i_2}x_{i_3}\cdots x_{i_l}(x_k - y_k)$ vanishes on $L$. Then the 1-form $dh$ has zero algebraic restriction to $L$. We have $dh = \xi_2 + \hat{\gamma}$, where

$$\begin{aligned}\hat{\gamma} = \quad & y_{i_2}x_{i_3}\cdots x_{i_l}(x_k-y_k)dy_{i_1} + y_{i_1}x_{i_3}\cdots x_{i_l}(x_k-y_k)dy_{i_2} \\ + \quad & \textstyle\sum_{u=3}^{l} y_{i_1}y_{i_2}x_{i_3}\cdots x_{i_{u-1}}x_{i_{u+1}}\cdots x_{i_l}(x_k-y_k)dx_u.\end{aligned}$$

Clearly $\hat{\gamma}$ has zero algebraic restriction to $L$. Then $\xi_2$ has zero algebraic restriction to $L$. The proof that 1-forms of the type $\rho$ and $\xi_1$ has zero algebraic restriction to $L$ follows from the fact that $\xi_2$ has zero algebraic restriction to $L$. $\square$

The following result provides a finite set of generators of $[\Lambda^{2,\mathrm{cl}}_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$.

**Proposition 4.5.** *A finite set of generators of $[\Lambda^{2,cl}_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$ is given by:*

- ***Degree 2:*** $[dx_i \wedge dy_j - dy_i \wedge dx_j]_L,\ \ 1 \le i \le j \le n;$
- ***Degree 3:*** $[d(x_iy_j)\wedge dx_k - d(y_iy_j)\wedge dx_k]_L,\ \ [d(x_iy_j)\wedge dx_k - d(x_iy_j)\wedge dy_k]_L,\ \ 1 \le i \le j \le n,\ \ 1 \le k \le n;$
- ***Degree 4:*** $[d(x_ix_jy_k)\wedge dx_l - d(y_iy_jx_k)\wedge dy_l]_L,\ \ 1 \le i \le j \le k \le n,\ \ 1 \le l \le n.$

*Proof.* According to Propositions 4.2 and 4.3, the derivatives of the generators of $[\Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$ generate $[\Lambda^{2,\mathrm{cl}}_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$. Therefore it is sufficient to verify that the algebraic restrictions represented by

- $x_idy_j - y_idx_j,\ \ 1 \le i \le j \le n$
- $x_iy_jdx_k - y_iy_jdx_k,\ \ x_iy_jdx_k - x_iy_jdy_k\ \ 1 \le i \le j \le n, 1 \le k \le n$
- $x_ix_jy_kdx_l - y_iy_jx_kdy_l,\ \ 1 \le i \le j \le k \le n, 1 \le l \le n$

generate $[\Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$. According to Proposition 4.1, we fix a degree and find generators for this fixed degree. Since the calculation is similar for each degree, we find a set of homogeneous generators of degree 4. The homogeneous 1-forms vanishing on $TL_{\mathrm{reg}}$ of degree 4 are generated by:

- $x_ix_jx_kdy_l - x_ix_jy_kdy_l$
- $x_ix_jx_kdy_l - x_iy_jx_kdy_l$
- $x_ix_jx_kdy_l - y_ix_jx_kdy_l$
- $x_ix_jx_kdy_l - x_iy_jy_kdy_l$
- $x_ix_jx_kdy_l - y_ix_jy_kdy_l$
- $x_ix_jx_kdy_l - y_iy_jx_kdy_l$
- $x_ix_jx_kdy_l - x_ix_jy_kdx_l$
- $x_ix_jx_kdy_l - x_iy_jx_kdx_l$
- $x_ix_jx_kdy_l - y_ix_jx_kdx_l$
- $x_ix_jx_kdy_l - x_iy_jy_kdx_l$
- $x_ix_jx_kdy_l - y_ix_jy_kdx_l$
- $x_ix_jx_kdy_l - y_iy_jx_kdx_l$
- $x_ix_jx_kdy_l - y_iy_jy_kdx_l$

where $i, j, k, l \in \{1, \ldots, n\}$. Adding a zero algebraic restriction to $L$ of the form $[h(x,y)dx_l]_L$ and $[h(x,y)dy_l]_L$, where $h$ is a polynomial vanishing on $L$, we reduce the generators of degree 4 to the following:

- $[x_ix_jx_kdy_l - x_ix_jy_kdy_l]_L$
- $[x_ix_jx_kdy_l - x_iy_jy_kdx_l]_L$
- $[x_ix_jx_kdy_l - y_iy_jy_kdx_l]_L$

where $i, j, k, l \in \{1, \dots, n\}$. Note that $d(x_ix_jx_ky_l - x_ix_jy_ky_l) \in \mathscr{A}_0^1(L, (\mathbb{R}^{2n}, 0))$. Moreover,

$$\begin{aligned} d(x_ix_jx_ky_l - x_ix_jy_ky_l) = \ & (x_jx_ky_l - x_jy_ky_l)dx_i + (x_ix_ky_l - x_iy_ky_l)dx_j + \\ & x_ix_jy_ldx_k - x_ix_jy_ldy_k + (x_ix_jx_k - x_ix_jy_k)dy_l. \end{aligned}$$

Then the algebraic restrictions of the type $[x_ix_jx_kdy_l - x_ix_jy_kdy_l]_L$ are generated by algebraic restrictions of the type $[x_ix_jy_kdy_l - y_iy_jx_kdx_l]_L$, $i, j, k, l \in \{1, \dots, n\}$. Similarly, $[x_ix_jx_kdy_l - x_iy_jy_kdx_l]_L$ and $[x_ix_jx_kdy_l - y_iy_jy_kdx_l]_L$ are generated by $[x_ix_jy_kdy_l - y_iy_jx_kdx_l]_L$, $i, j, k, l \in \{1, \dots, n\}$.

The algebraic restrictions

$$[x_ix_jy_kdx_l - y_iy_jx_kdy_l]_L, \ \ 1 \leq i \leq j \leq k \leq n, \ 1 \leq l \leq n,$$

generate the set algebraic restrictions of the 1-forms homogeneous of degree 4 in $[\Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})]_L$ since for all permutation of the indices $i, j, k$ the algebraic restrictions of the type $[x_ix_jy_kdx_l - y_iy_jx_kdy_l]_L$ are the same. □

**Definition 4.6.** *A germ of vector field $\eta$ on $(\mathbb{R}^m, 0)$ is liftable over a multigerm $F = \{F_1, \dots, F_s\} : (\mathbb{R}^k, S) \to (\mathbb{R}^m, 0)$ if there exist germs of vector fields $\xi_1, \dots, \xi_s$ on $(\mathbb{R}^k, 0)$ such that*

$$dF_i \circ \xi_i = \eta \circ F_i, \ \ i = 1, \dots, s.$$

*We denote the set of the germs of liftable vector fields over $F$ by* $\mathrm{Lift}(F)$*.*

Consider the multigerm $F : \{F_1, F_2, F_3\} : (\mathbb{R}^n, 0) \to (\mathbb{R}^{2n}, 0)$ defined by $F_1(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0)$, $F_2(y_1, \dots, y_n) = (0, \dots, 0, y_1, \dots, y_n)$ and $F_3(z_1, \dots, z_n) = (z_1, \dots, z_n, z_1, \dots, z_n)$.

**Proposition 4.7.** *The germs of liftable vector fields over $F$ are of the form $\sum_{i=1}^n \left(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}\right)$, where $X_i \in \langle x_1, \dots, x_n\rangle$, $Y_i \in \langle y_1, \dots, y_n\rangle$ and $X_i - Y_i \in \langle x_1 - y_1, \dots, x_n - y_n\rangle$, $i = 1, \dots, n$.*

*Proof.* Let $\eta = \sum_{i=1}^n \left( X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i} \right)$ be a germ of a vector field on $(\mathbb{R}^{2n}, 0)$ where $X_i, Y_i$ are as above. Consider the germs of the vector fields $\xi_1(x) = \sum_{i=1}^n X_i(x,0)\frac{\partial}{\partial x_i}$, $\xi_2(y) = \sum_{i=1}^n Y_i(0,y)\frac{\partial}{\partial y_i}$ and $\xi_3(z) = \sum_{i=1}^n X_i(z,z)\frac{\partial}{\partial z_i}$. Clearly $dF_i \circ \xi_i = \eta \circ F_i$, $i = 1,2,3$. Then $\eta$ is liftable over $F$.

Let $W = \sum_{i=1}^n \left( U_i \frac{\partial}{\partial x_i} + V_i \frac{\partial}{\partial y_i} \right) \in \mathrm{Lift}(F)$. Then there exist $\rho_1, \rho_2, \rho_3$ germs of vector fields on $\mathbb{R}^n$ such that $W \circ F_i = dF_i \circ \rho_i$, $i = 1,2,3$. Writing $\rho_1(x) = \sum_{j=1}^n \rho_1^j(x)\frac{\partial}{\partial x_j}$ for some germs of functions $\rho_1^j$, $j = 1, \dots, n$, one has

$$dF_1(x) \cdot \rho_1(x) = (\rho_1^1(x), \dots, \rho_1^n(x), 0, \dots, 0).$$

Then $V_i \in \langle y_1, \dots, y_n \rangle$, for all $i \in \{1, \dots, n\}$. Analogously we have $U_i \in \langle x_1, \dots, x_n \rangle$ and $U_i - V_i \in \langle x_1 - y_1, \dots, x_n - y_n \rangle$, for all $i = 1, \dots, n$. □

The next result establishes a relation between liftable and tangent vector fields.

**Proposition 4.8.** *If $\eta \in \mathrm{Lift}(F)$ then $\eta$ is tangent to $L$.*

*Proof.* Let $h$ be a germ of function vanishing on $L$. There exist germs of vector fields $\xi_i$ such that $\eta \circ F_i = dF_i \circ \xi_i$, $i = 1,2,3$. If $p \in (\mathbb{R}^n, 0)$ then

$$\begin{aligned}(dh \circ \eta)(F_i(p)) &= dh(F_i(p)) \cdot \eta(F_i(p)) = dh(F_i(p)) \cdot dF_i(p) \cdot \xi_i(p) \\ &= d(h \circ F_i)(p) \cdot \xi_i(p) = 0,\end{aligned}$$

since $h \circ F_i \equiv 0$ on $(\mathbb{R}^n, 0)$. □

**Proposition 4.9.** *Let $\eta \in \mathrm{Lift}(F)$ and let $\theta$ be a $k$-form with zero algebraic restriction to $L$. Then $\mathcal{L}_\eta \theta$ has zero algebraic restriction to $L$.*

*Proof.* It follows from the fact that $\eta$ is tangent to $L$ and $\mathcal{L}_\eta(d\beta) = d(\mathcal{L}_\eta \beta)$, for all $(k-1)$-form $\beta$. □

**Proposition 4.10.** *Let $\sigma \in \Lambda^{2,cl}_{\mathrm{reg}}(\mathbb{R}^{2n})$ and $\eta \in \mathrm{Lift}(F)$ then $\mathcal{L}_\eta \sigma \in \Lambda^{2,cl}_{\mathrm{reg}}(\mathbb{R}^{2n})$.*

*Proof.* By Proposition 4.2 there exists a 1-form $\gamma \in \Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})$ such that $\sigma = d\gamma$. Thus, $\mathcal{L}_\eta \sigma = \mathcal{L}_\eta d\gamma = d\mathcal{L}_\eta \gamma$. By Proposition 4.3, it is sufficient to prove that $\mathcal{L}_\eta \gamma$ vanishes on $TL_{\mathrm{reg}}$.

Let $\gamma = \sum_{j=1}^n (f_j(x,y)dx_j + g_j(x,y)dy_j) \in \Lambda^1_{\mathrm{reg}}(\mathbb{R}^{2n})$ and let $X_j, Y_j$ be germs of functions, $j = 1, \dots, n$, such that $\eta = \sum_{j=1}^n \left( X_j \frac{\partial}{\partial x_j} + Y_j \frac{\partial}{\partial y_j} \right)$. As $\gamma$ vanishes on

$TL_{\rm reg}$ we have $f_j(x,0)=0$, $g_j(0,y)=0$ and $(f_j+g_j)(z,z)=0$. Then

$$\mathcal{L}_\eta\gamma=\sum_{i,j=1}^n\left(\frac{\partial(f_jX_j)}{\partial x_i}dx_i+\frac{\partial(f_jX_j)}{\partial y_i}dy_i+\frac{\partial(g_jY_j)}{\partial x_i}dx_i+\frac{\partial(g_jY_j)}{\partial y_i}dy_i\right)$$

$$+\sum_{i,j=1}^n\left(\frac{\partial f_j}{\partial x_i}(X_idx_j-X_jdx_i)+\frac{\partial f_j}{\partial y_i}(Y_idx_j-X_jdy_i)\right)$$

$$+\sum_{i,j=1}^n\left(\frac{\partial g_j}{\partial x_i}(X_idy_j-Y_jdx_i)+\frac{\partial g_j}{\partial y_i}(Y_idy_j-Y_jdy_i)\right).$$

It follows from Proposition 4.7 that $\mathcal{L}_\eta\gamma$ vanishes on $TL_1$ and $TL_2$. The restriction of $\mathcal{L}_\eta\gamma$ to $TL_3$ is zero since

$$\mathcal{L}_\eta\gamma|_{L_3}=\sum_{i,j=1}^n\left(\frac{\partial(f_jZ_j)}{\partial z_i}(z,z)dz_i+\frac{\partial(g_jZ_j)}{\partial z_i}(z,z)dz_i\right)$$

$$+\sum_{i,j=1}^n\left(\frac{\partial f_j}{\partial z_i}(z,z)(Z_idz_j-Z_jdz_i)+\frac{\partial g_j}{\partial z_i}(z,z)(Z_idz_j-Z_jdz_i)\right)$$

$$=\sum_{i,j=1}^n\left(\frac{\partial((f_j+g_j)Z_j)}{\partial z_i}(z,z)dz_i+\frac{\partial(f_j+g_j)}{\partial z_i}(z,z)(Z_idz_j-Z_jdz_i)\right)$$

where $Z_j(z)=X_j(z,z)=Y_j(z,z)$, $j=1,\dots,n$. $\square$

Let $a$ be an algebraic restriction represented by a symplectic form vanishing on $TL_{\rm reg}$. Due to Proposition 4.5, $a$ has a symplectic representative $\sigma=\sigma^{(2)}+\sigma^{(3)}+\sigma^{(4)}$

where

$$\sigma^{(2)} = \sum_{1\leq i\leq j\leq n} a_{ij}(dx_i \wedge dy_j - dy_i \wedge dx_j)$$

$$\sigma^{(3)} = \sum_{\substack{1\leq i\leq j\leq n,\\ 1\leq k\leq n}} b^{(1)}_{ijk}(d(x_iy_j)\wedge dx_k - d(y_iy_j)\wedge dx_k) +$$

$$\sum_{\substack{1\leq i\leq j\leq n,\\ 1\leq k\leq n}} b^{(2)}_{ijk}(d(x_iy_j)\wedge dx_k - d(x_iy_j)\wedge dy_k)$$

$$\sigma^{(4)} = \sum_{\substack{1\leq i\leq j\leq k\leq n,\\ 1\leq l\leq n}} c_{ijkl}(d(x_ix_jy_k)\wedge dx_l - d(y_iy_jx_k)\wedge dy_l).$$

where $a_{ij}, b^{(1)}_{ijk}, b^{(2)}_{ijk}, c_{ijkl} \in \mathbb{R}$. Note that $\sigma(0)$ is represented by the matrix

$$M = \begin{bmatrix} 0 & C \\ -C & 0 \end{bmatrix},$$

where $C = (c_{ij}) \in \mathrm{GL}(n, \mathbb{R})$ is defined by $\begin{cases} c_{ij} = a_{ij},\ i<j \\ c_{ij} = a_{ji},\ i>j \\ c_{ii} = 2a_{ii},\ i = 1,\dots,n \end{cases}$.

**Proposition 4.11.** *The algebraic restriction $[\sigma]_L$ is diffeomorphic to $[\sigma^{(2)}]_L$.*

The proof of Proposition 4.11 follows from the next lemma.

**Lemma 4.12.** (i) *The algebraic restriction $[\sigma]_L$ is diffeomorphic to $[\sigma^{(2)} + \theta]_L$, where $\theta$ is a homogeneous 2-form of degree 4 vanishing on $TL_{\mathrm{reg}}$.*
(ii) *The algebraic restriction $[\sigma^{(2)} + \theta]_L$ is diffeomorphic to $[\sigma^{(2)}]_L$.*

*Proof.* We prove only the item (i) since the proof of (ii) is very similar. We use the Moser homotopy method. Let

$$\sigma^{(4)}_t = \sum_{\substack{1\leq i\leq j\leq k\leq n,\\ 1\leq l\leq n}} f_{ijkl}(t)(d(x_ix_jy_k)\wedge dx_l - d(y_iy_jx_k)\wedge dy_l),$$

where $f_{ijkl} : [0,1] \to \mathbb{R}$ are germs of functions with $f_{ijkl}(0) = c_{ijkl}$, $1 \leq i \leq j \leq k \leq n$ and $1 \leq l \leq n$. Let $\sigma_t = \sigma^{(2)} + (1-t)\sigma^{(3)} + \sigma^{(4)}_t$. Suppose that there exists $\Phi_t : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$, $t \in [0,1]$, a family of local symmetries of $L$ such that

$$\Phi_t^*[\sigma_t]_L = [\sigma]_L \ \text{ and } \ \Phi_0 = Id. \tag{4.0.1}$$

Differentiating (4.0.1) on $t$ we obtain

$$[\mathcal{L}_{\eta_t}\sigma_t]_L = \left[\sigma^{(3)} - \frac{d\sigma_t^{(4)}}{dt}\right]_L$$

where $\eta_t$ is obtained from the equation $d\Phi_t/dt = \eta_t \circ \Phi_t$.

According to Propositions 3.8 and 4.10, if $\eta_t \in \mathrm{Lift}(F)$ is homogeneous of degree 1 then

$$\mathcal{L}_{\eta_t}\sigma^{(2)} = \sigma^{(3)}.$$

We look for a germ of vector field $\eta_t$ satisfying

$$i_{\eta_t}\sigma^{(2)} = \sum_{\substack{1\le i\le j\le n,\\ 1\le k\le n}} (b^{(1)}_{ijk}(x_iy_jdx_k - y_iy_jdx_k) + b^{(2)}_{ijk}(x_iy_jdx_k - x_iy_jdy_k)).$$

If $\eta_t = \displaystyle\sum_{i=1}^{n}\left(X_i(t,x,y)\frac{\partial}{\partial x_i} + Y_i(t,x,y)\frac{\partial}{\partial y_i}\right)$ then

$$i_{\eta_t}\sigma^{(2)} = \sum_{i,j=1}^{n} e_{ij}(X_idy_j - Y_idx_j),$$

where $\begin{cases} e_{ij} = a_{ij},\ i<j \\ e_{ij} = a_{ji},\ i>j \\ e_{ii} = 2a_{ii},\ i = 1,\dots,n \end{cases}$ and $E = (e_{ij}) \in \mathrm{GL}(n,\mathbb{R})$. Therefore

$$\sum_{i,j=1}^{n} e_{ij}(X_idy_j - Y_idx_j) = \sum_{\substack{1\le i\le j\le n,\\ 1\le k\le n}} (b^{(1)}_{ijk}(x_iy_jdx_k - y_iy_jdx_k) + b^{(2)}_{ijk}(x_iy_jdx_k - x_iy_jdy_k)).$$

We have the following system

$$\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} X_1 \\ \vdots \\ X_n \\ Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \sum_{1\le i\le j\le n}\begin{pmatrix} -b^{(2)}_{ij1}x_iy_j \\ \vdots \\ -b^{(2)}_{ijn}x_iy_j \\ -b^{(1)}_{ij1}(x_iy_j - y_iy_j) - b^{(2)}_{ij1}x_iy_j \\ \vdots \\ -b^{(1)}_{ijn}(x_iy_j - y_iy_j) - b^{(2)}_{ijn}x_iy_j \end{pmatrix}.$$

Let $W = (w_{ij}) \in \mathrm{GL}(n, \mathbb{R})$ be the inverse matrix of $E$. The solution of the system is given by

$$\begin{aligned}
X_1 &= -\sum_{1\le i\le j\le n} (w_{11}(b^{(2)}_{ij1}x_ix_j) + \cdots + w_{1n}(b^{(2)}_{ijn}x_ix_j)) \\
&\vdots \\
X_n &= -\sum_{1\le i\le j\le n} (w_{n1}(b^{(2)}_{ij1}x_ix_j) + \cdots + w_{nn}(b^{(2)}_{ijn}x_ix_j)) \\
Y_1 &= -\sum_{1\le i\le j\le n} (w_{11}(b^{(1)}_{ij1}(x_iy_j - y_iy_j) + b^{(2)}_{ij1}x_iy_j) + \cdots + \\
&\qquad w_{1n}(b^{(1)}_{ijn}(x_iy_j - y_iy_j) + b^{(2)}_{ijn}x_iy_j)) \\
&\vdots \\
Y_n &= -\sum_{1\le i\le j\le n} (w_{n1}(b^{(1)}_{ij1}(x_iy_j - y_iy_j) + b^{(2)}_{ij1}x_iy_j) + \cdots + \\
&\qquad w_{nn}(b^{(1)}_{ijn}(x_iy_j - y_iy_j) + b^{(2)}_{ijn}x_iy_j)).
\end{aligned}$$

By Proposition 4.7 the germ of vector field $\eta_t = \sum_{i=1}^n \left(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}\right)$ is liftable over $F$.

According to Proposition 3.8 and 4.10 we have that $\mathcal{L}_{\eta_t}\sigma^{(4)}_t$ is a closed 2-form homogeneous of degree 5 vanishing on $TL_{\mathrm{reg}}$. It follows from Proposition 4.5 that $\mathcal{L}_{\eta_t}\sigma^{(4)}_t$ has zero algebraic restriction to $L$.

We determine the germs of functions $f_{ijkl}$ by the ODEs

$$i_{\eta_t}(1-t)\sigma^{(3)} = -\sum_{\substack{1\le i\le j\le k\le n,\\ 1\le l\le n}} \frac{df_{ijkl}(t)}{dt}(x_ix_jy_kdx_l - x_ix_jy_kdy_l)$$

with the initial data $f_{ijk}(0) = c_{ijkl}$.

We prove that

$$\mathcal{L}_{\eta_t}(\sigma^{(2)} + (1-t)\sigma_{(3)}) = \sigma^{(3)} - \frac{d}{dt}\sigma^{(4)}_t.$$

Thus the family of diffeomorphisms $\Phi_t$ associated to $\eta_t$ preserves $L$ since $\eta_t$ is liftable over $F$ and $\Phi_t^*[\sigma_t]_L = [\sigma]_L$, $t \in [0,1]$. Therefore $[\sigma]_L$ is diffeomorphic to $[\sigma^{(2)} + \theta]_L$, where

$$\theta = \sum_{\substack{1\le i\le j\le k\le n,\\ 1\le l\le n}} \tilde{c}_{ijkl}(d(x_ix_jy_k) \wedge dx_l - d(x_ix_jy_k) \wedge dy_l).$$

where $\tilde{c}_{ijkl} = f_{ijkl}(1)$ □

**Proposition 4.13.** *Let $\Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ be a local symmetry of $L$. Then the germ of diffeomorphism $\Phi^{(1)}$ is a local symmetry of $L$.*

*Proof.* Let $j_i$ be a permutation of $\{1, 2, 3\}$ such that $\Phi(L_i) = L_{j_i}$, for $i = 1, 2, 3$. Let $p \in L_i$, for some $i \in \{1, 2, 3\}$. We can write $\Phi$ in its Taylor series as

$$\Phi = \Phi^{(1)} + \tilde{\Phi}$$

where $\tilde{\Phi}^{(1)} = 0$. As $L_i$ is a germ of a linear subspace of $\mathbb{R}^{2n}$ then $\Phi(tp)/t$ belongs to the linear subspace $\ell_{j_i}$ which contains the germ $L_{j_i}$, for all $t \in (0, 1]$. Then

$$\lim_{t\to 0} \frac{\Phi(tp)}{t} = \Phi^{(1)}(p) \in \ell_{j_i}.$$

Taking $p$ close to the origin we have $\Phi^{(1)}(p) \in L_{j_i}$. □

As a consequence of Propositions 4.11 and 4.13, the classification of the algebraic restrictions to $L$ of symplectic forms vanishing on $TL_{\mathrm{reg}}$ under the action of local symmetries of $L$ reduces to the classification of algebraic restrictions to $L$ of symplectic forms homogeneous of degree 2 vanishing on $TL_{\mathrm{reg}}$ under the action of linear local symmetries of $L$. Since every 2-form does not have zero algebraic restriction to $L$ we have the following result:

**Proposition 4.14.** *Let $\sigma_1, \sigma_2$ be two symplectic forms vanishing on $TL_{\mathrm{reg}}$. Then $[\sigma_1]_L$ is diffeomorphic to $[\sigma_2]_L$ if and only if there exists a linear local symmetry of $L$ $\Psi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ such that $\Psi^*\sigma_2^{(2)} = \sigma_1^{(2)}$.*

## 5. Symplectic classification of transversal Lagrangian stars

One can get the following result by direct calculation.

**Proposition 5.1.** *Let $\Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ be a linear local symmetry of $L$. Then $\Phi$ is represented by one of the following matrices:*

1. $\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$
2. $\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}$
3. $\begin{bmatrix} B & 0 \\ B & -B \end{bmatrix}$
4. $\begin{bmatrix} B & -B \\ 0 & -B \end{bmatrix}$
5. $\begin{bmatrix} 0 & B \\ -B & B \end{bmatrix}$
6. $\begin{bmatrix} B & -B \\ B & 0 \end{bmatrix}$

*where $B \in \mathrm{GL}(n, \mathbb{R})$.*

Let $\sigma$ be a symplectic form homogeneous of degree 2 vanishing on $TL_{\mathrm{reg}}$. According to Proposition 4.5, $\sigma$ is written as $\sigma = \sum_{i,j=1}^{n} a_{ij} dx_i \wedge dy_j$, where $a_{ij} = a_{ji}$, $i, j = 1, \dots, n$. For all $p \in (\mathbb{R}^{2n}, 0)$, the bilinear form $\sigma(p) : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ has the representing matrix

$$W = \begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix},$$

where $A = (a_{ij})$.

Let $\Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ be a linear local symmetry of $L$ represented by the matrix of the type 1 of Proposition 5.1. The pullback of $\sigma$ by $\Phi$ is given by

$$(1) \begin{bmatrix} 0 & B^T AB \\ -B^T AB & 0 \end{bmatrix}.$$

For the other cases described in Proposition 5.1, the pullback has one of the followings representations

$$(2) \begin{bmatrix} 0 & -B^T AB \\ B^T AB & 0 \end{bmatrix}$$
$$(3) \begin{bmatrix} 0 & -B^T AB \\ B^T AB & 0 \end{bmatrix}$$
$$(4) \begin{bmatrix} 0 & -B^T AB \\ B^T AB & 0 \end{bmatrix}$$
$$(5) \begin{bmatrix} 0 & B^T AB \\ -B^T AB & 0 \end{bmatrix}$$
$$(6) \begin{bmatrix} 0 & B^T AB \\ -B^T AB & 0 \end{bmatrix}.$$

**Definition 5.2.** *Two matrices $A, B \in M(n, \mathbb{R})$ are congruent if there exists a matrix $P \in \mathrm{GL}(n, \mathbb{R})$ such that*

$$A = P^T BP.$$

The problem of classification of algebraic restrictions to $L$ of symplectic forms homogeneous of degree 2 vanishing on $TL_{\mathrm{reg}}$ under the action of linear local symmetries of $L$ is equivalent to the problem of classification of invertible symmetric matrices under the action of congruence. For such classification we use the Sylvester's Law of Inertia (see [R]).

For each $s \in \{0, \dots, n\}$, define $a_s = [\omega_s]_L$ where

$$\omega_s = dx_1 \wedge dy_1 + \cdots + dx_s \wedge dy_s - dx_{s+1} \wedge dy_{s+1} - \cdots - dx_n \wedge dy_n.$$

**Proposition 5.3.** *Let $\sigma$ be a symplectic form on $(\mathbb{R}^{2n}, 0)$ vanishing on $TL_{\mathrm{reg}}$. Then $[\sigma]_L$ is diffeomorphic to one and only one algebraic restriction of the type $a_s$, for some $s \in \{0, \dots, n\}$ with $s \leq \frac{n}{2}$.*

*Proof.* According to Proposition 4.14, it is sufficient to verify that for a symplectic form $\sigma$ homogeneous of degree 2 vanishing on $TL_{\mathrm{reg}}$ there exists a linear local symmetry of $L$ $\Psi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ such that $\Psi^*\sigma = \omega_s$, for a unique $s \in \{0, \dots, n\}$ such that $s \leq \frac{n}{2}$. According to Proposition 4.5, for all $p \in (\mathbb{R}^{2n}, 0)$ the bilinear form $\sigma(p) : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ is represented by the matrix

$$W = \begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix},$$

where $A \in \mathrm{GL}(n, \mathbb{R})$ is symmetric.

Due to Sylvester's Law of Inertia, there exists $B \in \mathrm{GL}(n, \mathbb{R})$ such that $B^T AB$ is equal to one and only one of the matrices $Z_u$ for some $u \in \{0, \dots, n\}$ where

$$Z_u = \begin{bmatrix} Id_u & 0 \\ 0 & -Id_{n-u} \end{bmatrix}.$$

If $u \leq \frac{n}{2}$ consider $\Phi$ the linear local symmetry of $L$ represented by

$$\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}.$$

The bilinear form $\Phi^*\sigma(p)$ is represented by

$$\begin{bmatrix} 0 & Z_u \\ -Z_u & 0 \end{bmatrix}.$$

Then $\Phi^*\sigma = \sum_{i=1}^{u} dx_i \wedge dy_i - \sum_{i=u+1}^{n} dx_i \wedge dy_i = \omega_u$

If $u > \frac{n}{2}$ consider $\Psi$ the linear local symmetry of $L$ represented by

$$\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}.$$

Thus $\Psi^*\sigma(p)$ is represented by

$$\begin{bmatrix} 0 & -Z_u \\ Z_u & 0 \end{bmatrix}.$$

Due to Sylvester's Law of Inertia, there exists $C \in \mathrm{GL}(n, \mathbb{R})$ such that $C^T(-Z_u)C = Z_{n-u}$. Let $H$ be a linear local symmetry of $L$ represented by

$$\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}.$$

We have

$$H^*\Psi^*\sigma = \sum_{i=1}^{n-u} dx_i \wedge dy_i - \sum_{i=n-u+1}^{n} dx_i \wedge dy_i = \omega_{n-u}.$$

Then the orbits of algebraic restrictions to $L$ of symplectic forms have symplectic representatives $\omega_s$, for $s \leq \frac{n}{2}$.

It remains only to prove that the orbits of $a_1, \ldots, a_l$ are disjoints, where $l$ is the biggest integer such that $l \leq \frac{n}{2}$. Let $u, v \in \{0, \ldots, n\}$, $u, v \leq \frac{n}{2}$ and $u \neq v$. The forms $\omega_u(p)$ and $\omega_v(p)$ are represented by:

$$\omega_u(p) = \begin{bmatrix} 0 & Z_u \\ -Z_u & 0 \end{bmatrix} \text{ and } \omega_v(p) = \begin{bmatrix} 0 & Z_v \\ -Z_v & 0 \end{bmatrix}.$$

Note that the signatures of $Z_v$ and $-Z_v$ are distinct of the signature of $Z_u$. According to Sylvester's Law of Inertia, the matrix $Z_u$ is not congruent neither to $Z_v$ nor to $-Z_v$. It follows from Proposition 5.1 that there is no linear local symmetry of $L$ $T : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ such that $T^*a_u = a_v$. Therefore the orbits represented by $a_u$ and $a_v$ are disjoints. □

Now we have the elements to obtain the symplectic classification of transversal Lagrangian stars in $((\mathbb{R}^{2n}, \omega), 0)$.

*Proof of Theorem 2.3.* Due to Proposition 5.3, the orbits of the algebraic restrictions to $L$ of symplectic forms vanishing on $TL_{\mathrm{reg}}$ are $a_1 = [\omega_1]_L, \ldots, a_u = [\omega_u]_L$, where $u$ is the biggest integer that satisfies $u \leq \frac{n}{2}$. Let $s$ be a positive integer such that $s \leq u$. Consider the germ of diffeomorphism $\Phi_s : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ defined by $\Phi_s(x, y) = (x, y_1, \ldots, y_s, -y_{s+1}, \ldots, -y_n)$. Note that $\Phi_s^*\omega_s = \omega$. Then the algebraic restrictions $a_1, \ldots, a_u$ are diffeomorphic respectively to

$$[\omega]_{\Phi_1^{-1}(L)}, \ldots, [\omega]_{\Phi_u^{-1}(L)}.$$

Let $E^s = (\{\Phi_s^{-1}(L_1), \Phi_s^{-1}(L_2), \Phi_s^{-1}(L_3)\}, 0)$.

Finally we apply Theorem 3.3 to obtain the normal forms of transversal Lagrangian stars $E^1, \ldots, E^u$. □

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